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A STABLE LEAST-SQUARES FINITE ELEMENT METHOD FOR NON-LINEAR HYPERBOLIC PROBLEMS

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SUMMARY

A class of stable least-square finite element methods for non-linear hyperbolic problems is developed and some exploratory studies made. The methods are based on modifying the L^2-norm of the residual and a related approximation to the H^1-norm of the residual. The effect of the additional terms in these residual functionals is to introduce a dissipative effect proportional to the solution gradient. This acts to stabilize the solution for non-linear hyperbolic problems which generate shocks. Numerical results for a one-dimensional nozzle and shock tube problem demonstrate the accuracy and stability of the method. Results are for an implicit scheme and calculations for linear, quadratic and cubic elements are given.

KEY WORDS Least squares Finite elements Non-linear Hyperbolic

INTRODUCTION

There have been several interesting studies of least-squares methods for solving partial differential equations. Eason[1] presents a review of some earlier work in this area. Recently, least-squares finite element methods have been constructed and applied to a variety of different problems with some success.[2–13] In previous studies[12, 13] we described a least-squares residual finite element method for hyperbolic problems. There it was demonstrated that this approach acts naturally in a manner similar to streamline upwinding and contains no 'free' parameters. For linear problems the method is unconditionally stable at all Courant numbers and the numerical results of its performance are very encouraging.

Our approach in these previous studies has been based on minimizing the residual in the L^2 norm. However, for high-speed compressible flow problems numerical experiments reveal that this approach fails due to the presence of a non-linear instability that becomes pronounced as a developing shock steepens. Similar difficulties have been observed for other finite element methods applied to this class of problems and various forms of artificial dissipation have been introduced to stabilize the methods. For example, Löhner *et al.*[14] and Oden *et al.*[15] employ Lapidus' artificial viscosity in their Taylor–Galerkin finite element method; Morgan *et al.*[16] examine 'flux-corrected transport'; Hughes and Mallet[17] adapt some ideas from recent high-resolution difference techniques into their Petrov–Galerkin finite element scheme; and Selmin and Quartapelle[18] add an artificial viscosity to the Taylor–Galerkin method of Donea. Hughes[19] has made several extensions to develop SUPG schemes and Wahlbin *et al.*[20] have provided related error estimates.

In the present paper we first describe the L^2-residual least-squares method for one-dimensional hyperbolic systems and later demonstrate the numerical instability. We then propose modifica-

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tions to the form of this functional based on the least-squares H^1-residual functional. The objective is to exert derivative control using ideas from multi-objective optimization theory to stabilize the approximate solution. This approach yields an additional numerical viscosity proportional to the solution gradient. Numerical results for nozzle and shock tube flows conclude the development. Our purpose here is to extend the least-squares approach and make some preliminary studies of its numerical accuracy to better understand this class of methods. No definitive comparison with other methods is attempted.

FORMULATION

L^2-residual scheme

Consider a first-order hyperbolic problem of the form

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=f. \tag{1}$$

For a given time step $\Delta t=t_{n+1}-t_n$, let us linearize the problem by setting $A_n=A(u_n)$ and then backward difference to obtain the implicit time-differenced problem

$$\frac{u_{n+1}-u_n}{\Delta t}+A_n\frac{\partial u_{n+1}}{\partial x}-f_{n+1}=0. \tag{2}$$

Introducing the L^2-norm of the residual R for admissible u_{n+1} in (2),

$$I(u_{n+1})=\int_\Omega R^2\,\mathrm{d}x=\int_\Omega\left(\frac{u_{n+1}-u_n}{\Delta t}+A_n\frac{\partial u_{n+1}}{\partial x}-f_{n+1}\right)^2\mathrm{d}x. \tag{3}$$

Taking variations with respect to u_{n+1} and setting $v=\delta u_{n+1}$, $\delta I=0$ implies

$$\int_\Omega\left(\frac{u_{n+1}-u_n}{\Delta t}+A_n\frac{\partial u_{n+1}}{\partial x}-f_{n+1}\right)\left(\frac{1}{\Delta t}+A_n\frac{\partial}{\partial x}\right)v\,\mathrm{d}x=0, \tag{4}$$

which evidently corresponds to a Petrov–Galerkin-type scheme with test function $v+\Delta t\,A_n\,\partial v/\partial x$. This approach introduces a numerical dissipation scheme of similar nature to that in Lax–Wendroff and Taylor–Galerkin methods as demonstrated in Carey and Jiang.[12] However, as in these methods, it is found that due to the non-linearity, computations for high-speed compressible flows become numerically unstable when shocks form. We present some representative numerical results for a shocked nozzle flow exhibiting the unstable behaviour later.

Weighted H^1-residual schemes

Since the non-linear instability becomes manifest as the solution gradient steepens during shock formation, this suggests that additional derivative control could provide a mechanism for stabilizing the solution. In the context of our least-squares residual formulation, this can be achieved in several ways; e.g. by including additional artificial viscosity terms in the functional, adding a penalty functional to constrain the gradients or considering alternative forms of the functional. For example, a natural modification is the weighted H^1-norm of R,

$$I=\int_\Omega\left[R^2+\beta\left(\frac{\partial R}{\partial x}\right)^2\right]\mathrm{d}x, \tag{5}$$

for the minimization problem, where β is a positive small-scale factor. We can view (5) simply as a multi-objective optimization function where the weight β is chosen to emphasize the principal objective function of interest.

Introducing the residual R of (2) into (5) and taking variations, the weak statement becomes

$$\int_{\Omega}\left\{\left[\left(1+\Delta t\,A_n\frac{\partial}{\partial x}\right)(u_{n+1}-u_n)+\Delta t\left(A_n\frac{\partial u_n}{\partial x}-f_{n+1}\right)\right]\left(1+\Delta t A_n\frac{\partial}{\partial x}\right)v\right.$$
$$+\beta\left[\left(\left(1+\Delta t\frac{\partial A_n}{\partial x}\right)\frac{\partial}{\partial x}+\Delta t\,A_n\frac{\partial^2}{\partial x^2}\right)(u_{n+1}-u_n)\right.$$
$$\left.\left.+\Delta t\left(\frac{\partial A_n}{\partial x}\frac{\partial u_n}{\partial x}+A_n\frac{\partial^2 u_n}{\partial x^2}-\frac{\partial f_{n+1}}{\partial x}\right)\right]\left[\left(\left(1+\Delta t\frac{\partial A_n}{\partial x}\right)\frac{\partial}{\partial x}+\Delta t A_n\frac{\partial^2}{\partial x^2}\right)v\right]\right\}\mathrm{d}x=0. \tag{6}$$

Introducing finite element expansions for u_n and u_{n+1}, together with the above modified least-squares approximations, yields a linear system of equations to be solved for the vector of nodal unknowns defining u_{n+1}. We seek to obtain a finite element solution with good shock resolution, but also suppress the numerical instability at the shock due to the increasing derivative. The additional functional acts as a control mechanism on the size of the derivative but should not be so strong that the solution is over-dissipative. Note, however, that the form in (6) contains second derivatives of u and v which implies that C^1 elements such as cubic splines are appropriate for a conforming scheme based on this higher-order statement. Since in practice we want to be able to employ simpler elements with C^0 bases, we consider two practical approximate forms based on (5) and (6). In the first we assume that the second-order derivative terms in (6) are of lesser importance and may be omitted. The resulting integral statement now becomes

$$\int_{\Omega}\left\{\left[\left(1+\Delta t\,A_n\frac{\partial}{\partial x}\right)(u_{n+1}-u_n)+\Delta t\left(A_n\frac{\partial u_n}{\partial x}-f_{n+1}\right)\right]\left(1+\Delta t A_n\frac{\partial}{\partial x}\right)v\right.$$
$$\left.+\beta\left[\left(1+\Delta t\frac{\partial A_n}{\partial x}\right)\frac{\partial}{\partial x}(u_{n+1}-u_n)+\Delta t\left(\frac{\partial A_n}{\partial x}\frac{\partial u_n}{\partial x}-\frac{\partial f_{n+1}}{\partial x}\right)\right]\left[\left(1+\Delta t\frac{\partial A_n}{\partial x}\right)\frac{\partial v}{\partial x}\right]\right\}\mathrm{d}x=0, \tag{7}$$

which involves an additional artificial dissipation contribution, $\beta(1+\Delta t\,A_n')^2 u_{n+1}' v'$, where the 'prime' indicates differentiation in x. This term corresponds to adding the stabilization functional $\beta\int(1+\Delta t\,A_n')^2(u_{n+1}')^2\,\mathrm{d}x$ to $\int R^2\,\mathrm{d}x$. Integrating by parts reveals that the corresponding numerical viscosity term in the associated Euler equation is

$$-\beta[(1+\Delta t\,A_n')^2 u_{n+1}']'.$$

That is, for $\beta\neq 0$ we have added an artificial dissipation with a term of order Δt proportional to A_n' and the proposed scheme is a specific form of artificial dissipation technique. As an example, in the familiar Burgers' equation $A_n'=u_n'$ so that, as u_n' grows locally and the shock steepens, proportional dissipation is added to stabilize the calculation. In the numerical experiments discussed later we have $0\leqslant\beta\ll 1$.

The second appropriate formulation based on (6) can be constructed using higher-degree elements in a C^0 non-conforming method. That is, for (5) we write the element accumulation

$$\tilde{I}=\sum_{e=1}^{E} I_e=\sum_{e=1}^{E}\left\{\int_{\Omega_e}\left[R^2+\beta\left(\frac{\partial R}{\partial x}\right)^2\right]\mathrm{d}x\right\} \tag{8}$$

and use a 'non-conforming' C^0 Lagrange finite element basis to construct a finite element approximation to the minimizer of $\tilde{I}$. For example, the contributions of the second derivatives in the element interiors can be calculated with non-conforming C^0 quadratic or cubic elements. Some representative results for this approximation are also given later.

The basic formulation above can be extended in several ways. The generalization to hyperbolic systems in one dimension is immediate—simply interpret u, f as vectors and A as a matrix (see the numerical examples following). A similar treatment can be made for problems in higher dimension and will be considered in a later study.

NUMERICAL RESULTS

Inviscid Burgers' equation

As an introductory example, we consider the inviscid Burgers' equation

$$u_t + uu_x = 0, \tag{9}$$

with initial data

$$u(x, 0) = g(x) \tag{10}$$

corresponding to an initial slant step.

The slant step steepens as time increases to form a 'shock.' In the absence of derivative control ($\beta = 0$), the calculations are non-linearly unstable in that, as the shock forms, local oscillations develop (Figure 1) that grow catastrophically. Similar problems arise from the Taylor–Galerkin approximation and the solution 'blows up' after $t = 0{\cdot}4$.[12] In Figure 2(a) we show the computed solution for scheme (7) at $t = 0{\cdot}4$ for computations on a uniform mesh of 100 linear elements with fixed time step $\Delta t = 0{\cdot}02$ and parameter $\beta = 0{\cdot}00001$. The numerical experiment is repeated in Figure 2(b) for scheme (8) on a uniform mesh of 50 quadratic elements with parameter $\beta = 0{\cdot}0001$. The derivative control stabilizes the procedure.

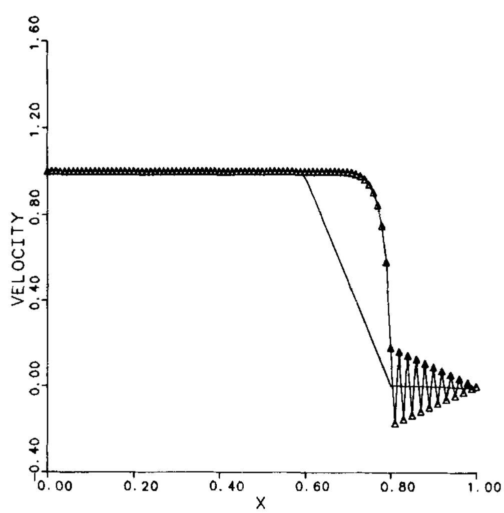

Figure 1. Oscillation of solution to Burgers' equation as shock forms for L^2-residual approach: 100 linear elements, $\Delta t = 0{\cdot}02$, $t = 0.2$

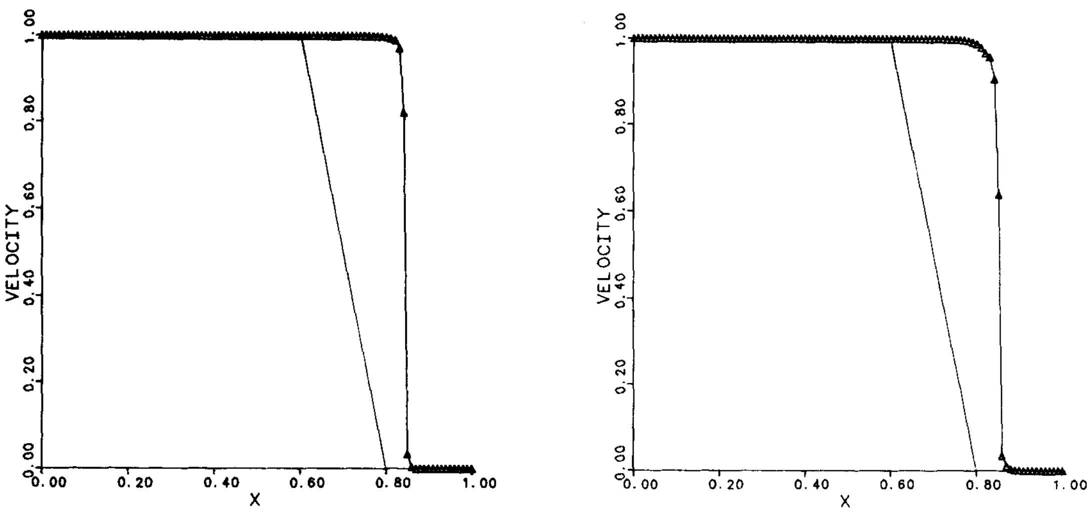

Figure 2. Solution to Burgers' equation from initial slant step: (a) 100 linear elements, $\Delta t=0{\cdot}02$, $t=0{\cdot}4$, $\beta=0{\cdot}00001$; (b) 50 quadratic elements, $\Delta t=0{\cdot}02$, $t=0{\cdot}4$, $\beta=0{\cdot}0001$

Isothermal flow in a nozzle

The equations governing isothermal flow in a nozzle constitute a system of the form (1) with

$$\mathbf{u}=\begin{bmatrix}\rho a\\ \rho a u\end{bmatrix},\quad \mathbf{A}=\begin{bmatrix}0 & 1\\ c^2-u^2 & 2u\end{bmatrix},\quad \mathbf{f}=\begin{bmatrix}0\\ \rho c^2\,\mathrm{d}a/\mathrm{d}x\end{bmatrix},\tag{11}$$

where ρ is the density, u is the velocity, the speed of sound c is normalized to unity and the cross-sectional area of the nozzle is

$$a=1{\cdot}0+(x-2{\cdot}5)^2/12{\cdot}5,\quad 0\leqslant x\leqslant 5.\tag{12}$$

In Figure 3 numerical results for u are shown for the case of subsonic inflow and outflow with an

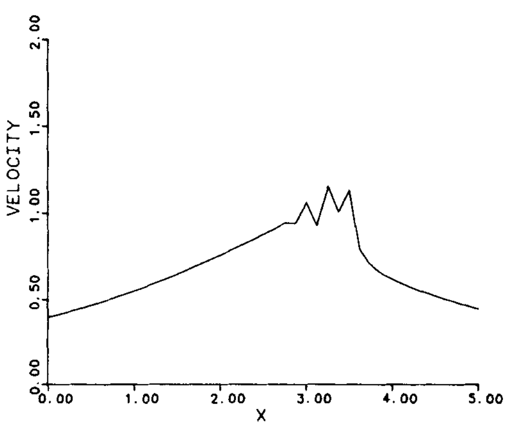

Figure 3. Deterioration of isothermal nozzle flow solution as shock forms for L^2-residual approach: $h=0{\cdot}125$, $\Delta t=0{\cdot}5$, $t=8{\cdot}5$; initial condition $\rho a u=1{\cdot}0$, $\rho=1{\cdot}493$

interior supersonic flow regime terminating in a shock, using the L^2-residual method and linear elements. As indicated previously, the shock begins to form but becomes unstable near $t=8\cdot5$ for calculation on a mesh with 40 uniform linear elements. Mesh refinement to 80 uniform linear elements does not remove or delay the onset of instability. Similar instabilities arise for calculations with more conventional Petrov–Galerkin and Taylor–Galerkin methods in the absence of a control mechanism such as that indicated here. Results for the first stabilized scheme (7) and linear elements are shown in Figure 4 for calculations with $h=0\cdot125$ once again and a time step $\Delta t=0\cdot5$, with $\beta=0\cdot0001$. The (essentially) steady-state solution is attained after 80 time steps. These results compare favourably with those of Hughes and Tezduyar[20] and of Löhner *et al.*[14] who use different artificial viscosity techniques as stated in the Introduction.

Shock tube problem

The one-dimensional Euler equations for compressible flow can be expressed as a system in the form (1) with

$$\mathbf{u}=\begin{bmatrix}\rho\\ u\\ p\end{bmatrix},\qquad \mathbf{A}=\begin{bmatrix}u & \rho & 0\\ 0 & u & \rho^{-1}\\ 0 & \gamma p & u\end{bmatrix},\qquad \mathbf{f}=\mathbf{0}, \tag{13}$$

where ρ is the density, u is the velocity, p is the pressure and γ is the ratio of specific heats. For the shock tube problem the initial data are

$$\mathbf{u}=\begin{bmatrix}1\cdot0\\ 0\cdot0\\ 1\cdot0\end{bmatrix}\quad \text{for } x\leqslant 0\cdot50,\qquad \mathbf{u}=\begin{bmatrix}0\cdot125\\ 0\cdot0\\ 0\cdot1\end{bmatrix}\quad \text{for } x>0\cdot50.$$

The objective is to compute accurate approximations to ρ, u and p at subsequent times as the shock and contact discontinuities propagate along the tube. This problem was computed by Sod[22] with finite difference techniques and subsequently investigated by Baker,[23] Löhner *et al.*[14] and Hughes and Mallet[17] using finite elements. The problem is solved here using the modified least-squares scheme (7) with a uniform mesh of 100 linear elements and fixed time step $\Delta t=0\cdot005$.

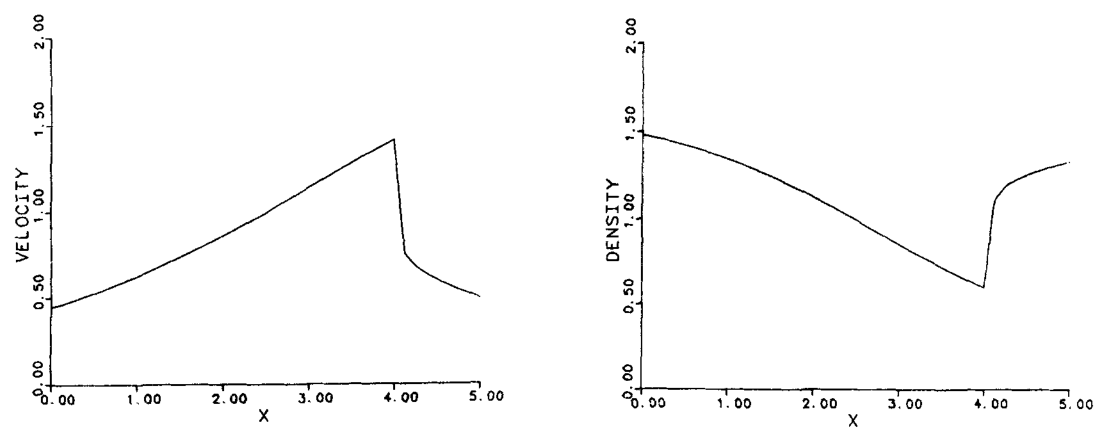

Figure 4. Stabilized solution at $t=40$ for isothermal flow in a nozzle using modified functionals and linear elements: $h=0\cdot125$, $\Delta t=0\cdot5$, $\beta=0\cdot0001$

The solution profiles for density, velocity and pressure at $t=0{\cdot}14$ are shown in Figure 5. There is moderate smearing of the 'fronts' and the solution is neither oscillatory nor unstable. We remark that these calculations with linear elements were stable for $\beta=0$. Finally we also include sample results for pressure at $t=0{\cdot}14$ in Figure 6 obtained for the 'non-conforming' approximation (8) based on C^0 linear, quadratic and cubic elements. For quadratic and cubic elements the method is unstable for $\beta=0$ but stable for small positive β. The linear case (Figure 6(a)) reduces to scheme (7). Increasing the degree of the element with this formulation appears to have little effect on accuracy since the error is dominated by the time integration error and, further, dissipation is proportional to the time step. Reducing the time step would improve accuracy but increase computation time.

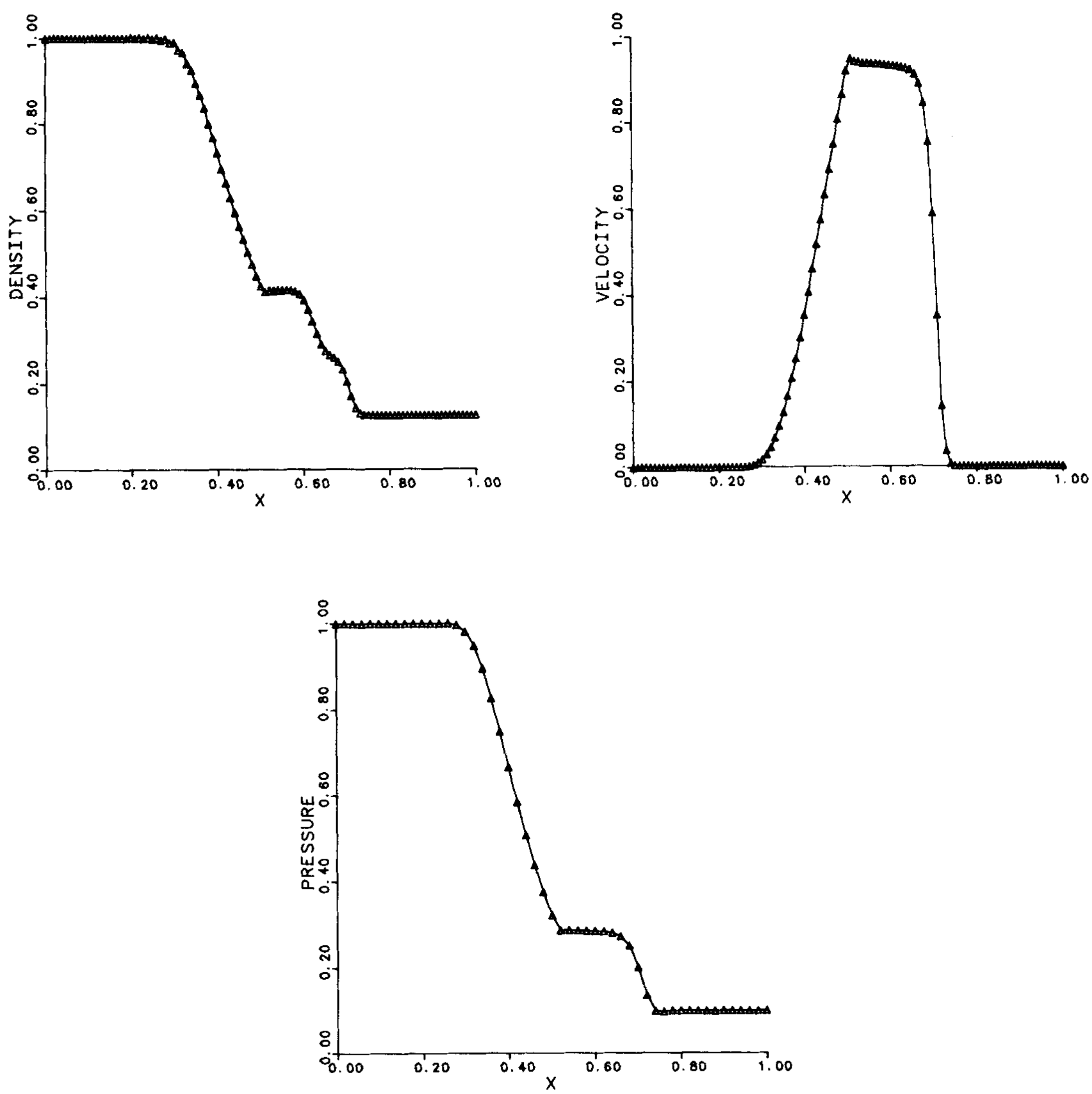

Figure 5. Density, velocity and pressure profiles for Sod's shock tube problem with $h=0{\cdot}01$, $\Delta t=0{\cdot}005$, $\beta=0{\cdot}0$ at $t=0{\cdot}14$ using linear elements

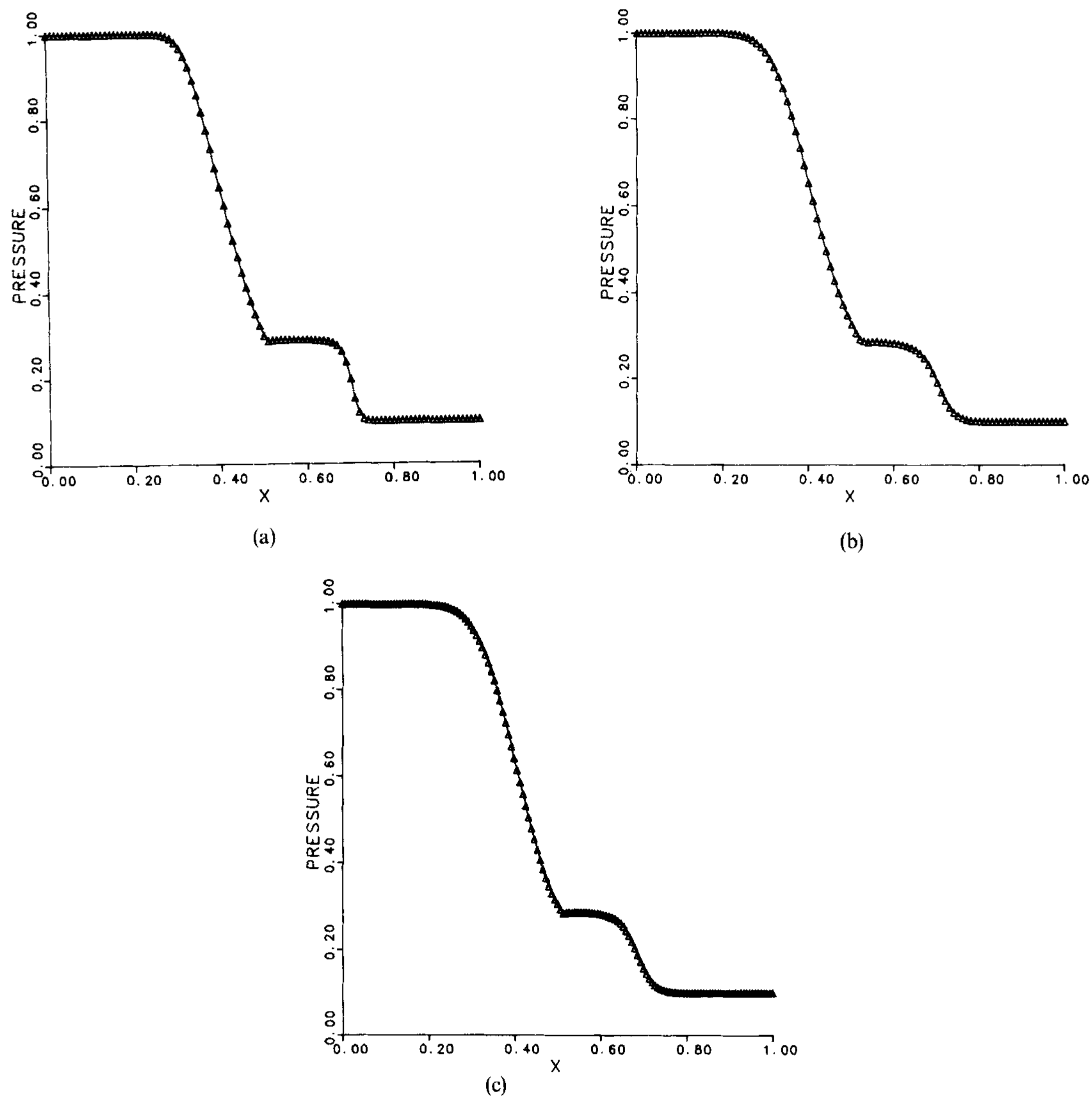

Figure 6. Results for Sod's problem using linear, quadratic and cubic elements: (a) 50 linear elements, $\Delta t = 0{\cdot}007$, $t = 0{\cdot}14$, $\beta = 0{\cdot}0$; (b) 50 quadratic elements, $\Delta t = 0{\cdot}007$, $t = 0{\cdot}14$, $\beta = 0{\cdot}00001$; (c) 50 cubic elements, $\Delta t = 0{\cdot}007$, $t = 0{\cdot}14$, $\beta = 0{\cdot}00001$

CONCLUDING REMARKS

A modified form of the least-squares finite element method is developed. The scheme appears stable for non-linear problems such as those associated with shocks in gas dynamics. In this presentation we have considered an implicit backward-difference time discretization of the locally linearized equations. Clearly other variants of the scheme are possible and some generalizations are given in Carey and Jiang.[12] We are also presently extending the scheme to two-dimensional Euler equation calculations for shocked compressible flows.[24]

The implicit schemes involve system solutions in each time step and consequently for two- and three-dimensional problems are expensive if time-accurate solutions are needed. For implicit linear schemes we have also shown[12] that the method is unconditionally stable at all Courant numbers and hence the scheme promises to be efficient for computing steady flow fields. This idea has also been receiving increasing attention as a viable strategy by researchers using other methods. Moreover, quasi-explicit schemes can also be constructed by lumping and transposing the implicit contribution of the algebraic system to the right side and iteratively correcting this term repeatedly with each time step. However, for large time steps the added dissipation near shocks will be significant and the approximation dissipative. Thus the stability is offset by loss of accuracy and this appears to be a principal detraction of the method at present.

Finally we remark that in the course of the present numerical studies we also examined some of the other artificial dissipation schemes used in Galerkin-based finite element algorithms and finite difference schemes, specifically those included in a Lapidus-type dissipation term and a TVD-type term. These were included in the least-squares approach via additional contributions to the L^2-residual functional. Results with these least-squares schemes were inferior to those obtained here using the dissipative control term involving the element residual derivatives.

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